NASR-49(07)

Reprinted from

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 4, No. 5, MAY 1963

Printed in U.S.A.

p622-633 refs

Three-Particle Scattering. I. Planar Case*

N64 10237

PP. 622-633

R. C. WHITTEN

8298002 Stanford Research Institute, Menlo Park, California

(Received 6 December 1962)

CODE-NONE

10237

A formal theory of three-particle scattering in a plane is developed using integral equation methods. Expressions for the scattering amplitude and cross section of elastic and inelastic collisions are derived. The effects of indistinguishability of the colliding particles are discussed.

(NASA Contract NASr-49(07))

AUTHOR (NASA CR-52223)

I. INTRODUCTION

THE theory of three-body scattering in its general case has obvious applications to ionic and chemical reactions; in spite of well-known difficulties it may even be possible to compute the rate constants for such simple cases as three-body electron attachment to hydrogen atoms. Such calculations can be approached via the standard method of collision cross sections, or alternatively, by means of an appropriate representation of the scattering operator.[1] Investigations of some aspects of three-body collisions have been carried out by Delves[2] and by Gallina and coworkers.[3] The latter have obtained the Green's function for three-particle scattering in three-dimensional space, but for S waves only.

Recent developments in the theory of generalized angular momentum representations[4,5] are admirably suited to the treatment of three-particle collisions. For the present we shall restrict our investigation to motion in a plane, emphasizing developments which can be utilized in the extension of our treatment of the problem to three dimensions.

Smith's formalism[4,5] for treating three-body collisions is expressed in terms of a generalized angular momentum operator $\mathbf{\Lambda}$, components of which are given by

$$_m\Lambda_{ij}^{kl} = {}_m\xi_i^k\, {}_m\pi_j^l - {}_m\xi_j^l\, {}_m\pi_i^k, \tag{1}$$

where m labels the order in which the particles are paired, e.g., 1 to 2, and 3 to the 1–2 pair, k and l are particle indices, and i and j denote the cartesian components of the appropriate position (momentum) vectors. The symbols $_m\xi^i$ and $_m\pi^i$ represent the "normalized" position and momentum of the ith and jth particle pairs in the center of mass system with coupling order m:

$$\begin{aligned}
{}_m\xi^1 &= d_m^{-1}(\mathbf{x}^{m+2} - \mathbf{x}^{m+1}),\\
{}_m\xi^2 &= d_m[\mathbf{x}^m - (m_{m+1} + m_{m+2})^{-1}\\
&\qquad \times (m_{m+1}\mathbf{x}^{m+1} + m_{m+2}\mathbf{x}^{m+2})],\\
{}_m\pi^1 &= \frac{\mu}{d_m}\left(-\frac{\mathbf{p}^{m+1}}{m_{m+1}} + \frac{\mathbf{p}^{m+2}}{m_{m+2}}\right),\\
{}_m\pi^2 &= \mu d_m\left(\frac{\mathbf{p}^m}{m_m} - \frac{\mathbf{p}^{m+1} + \mathbf{p}^{m+2}}{m_{m+1} + m_{m+2}}\right),
\end{aligned} \tag{2}$$

in which

$$\begin{aligned}
\mu^2 &= m_1m_2m_3/M \quad \text{(reduced mass)},\\
M &= m_1 + m_2 + m_3,\\
d_k^2 &= (m_k/\mu)(1 - m_k/M).
\end{aligned} \tag{3}$$

In this representation the center-of-mass position $\mathbf{X}$ and momentum $\mathbf{P}$ are given by

$$\mathbf{X} = \sum_{k=1}^{3} \frac{m_k}{M}\mathbf{x}^k, \qquad \mathbf{P} = \sum_{k=1}^{3}\mathbf{p}^k. \tag{4}$$

From the variables (2), one can form several independent dynamical variables related to angular momentum of which

$$\Lambda^2 = \tfrac{1}{2}\sum_{ijkl}|\Lambda_{ij}^{kl}|^2, \tag{5a}$$

$$\Sigma_t = \sum_i \Lambda_{ii}^{12}, \tag{5b}$$

$$L_1 = \Lambda_{12}^{11}, \tag{5c}$$

$$L_2 = \Lambda_{12}^{22}, \tag{5d}$$

$$L = L_1 + L_2, \tag{5e}$$

$$Y = L_1 - L_2, \tag{5f}$$

are the most important; they become operators upon making the appropriate quantum-mechanical replacements, e.g., $_m\pi_i^1 \to i\hbar\,(\partial/\partial_m\xi_i^1)$. Since the motion in the center-of-mass system embodies four degrees of freedom, a full description of the assembly

* This research was supported by the National Aeronautics and Space Administration and the National Science Foundation.

[1] F. T. Smith, J. Chem. Phys. 36, 250 (1962).

[2] L. M. Delves, Nucl. Phys. 9, 391 (1958–59); 20, 275 (1960).

[3] V. Gallina, P. Nata, L. Bianchi, and G. Viano, Nuovo Cimento 24, 835 (1962).

[4] F. T. Smith, Phys. Rev. 120, 1058 (1960).

[5] F. T. Smith, J. Math. Phys. 3, 735 (1962).

of particles must contain four and only four independent dynamical variables (exclusive of those associated with internal structure), one of which is conveniently taken as the kinetic energy K. Three useful choices of the other three are (A) Λ^2, L and Σ_t, (B) Λ^2, L and Y, and equivalently (C) Λ^2, L_1 and L_2. With the aid of raising and lowering operators for the appropriate dynamical variables, Smith[5] has constructed specific representations of the unitary transformations connecting representations (A) and (B); representations (B) and (C) differ only by a phase factor. The elements of the transformation are identical with the b coefficients contained in expressions (29) and (30) of reference 5. Representation (A) has been appropriately termed symmetric because all three particles are treated on a completely equal footing. On the other hand, representations (B) and (C) depend upon the order in which the particles are paired; hence they are not treated on an equal basis and we call them asymmetric. This can also be stated in terms of transformation properties under a change in the pairing order; this involves an orthogonal transformation of the ξ's and π's, called by Smith a kinematic rotation. Representation (A) is invariant under such a transformation while (B) and (C) are not.

One can attach the following physical significance to the dynamical variables L, L_1, L_2, Λ^2, and Σ_t. That of L, L_1, and L_2 is immediately apparent: they are respectively the total *ordinary* angular momentum, the ordinary angular momentum of particles 1 and 2 about their center of mass, and the ordinary angular momentum of particle 3 about the center of mass of particles 1 and 2. Actually we should add a prefix to L_1 and L_2 to denote the order of pairing of the particles, e.g., ${}_mL_1$. Smith[4] has related (classically) Λ^2 to the kinetic energy K in the center-of-mass system and a three-body "impact parameter" R by the equation

$$\Lambda^2 = 2\mu KR^2, \tag{6}$$

where R is defined as the minimum value of $\rho = [\sum_{i,j} (\xi_i^j)^2]^{\frac{1}{2}}$ on a straight-line trajectory. Hence Λ^2 is a measure of the tendency of the three particles to simultaneously pass through a given region. We now raise the question, how closely for a given value of R does the system approach the situation in which the particles are equidistant at the instant of time when $\rho = R$? In other words, when would $|\mathbf{x}^1 - \mathbf{x}^2| = |\mathbf{x}^2 - \mathbf{x}^3| = |\mathbf{x}^3 - \mathbf{x}^1|$? This question can best be answered if we first go to the one-dimensional case in which Σ_t and K are the only dynamical variables. The particles approach most closely when Σ_t is a minimum and coincide at some instant of time if $\Sigma_t = 0$. This interpretation of Σ_t can be readily carried over to the planar case; for a given value of Λ^2 the particles approach a three-body collision most closely for $\Sigma_t = 0$ and progressively less closely as Σ_t increases. Obviously, if $\Lambda^2 = 0$, we must have $\Sigma_t = 0$.

As Smith has demonstrated, the classical argument can readily be carried over into the quantum domain. When the eigenvalue of the square of the generalized angular momentum Λ^2 vanishes the particles are coincident at some instant of time and the eigenvalue of Σ_t, σ, must also vanish; this is indeed a consequence of the properties of $\mathbf{\Lambda}$. One can, of course, also characterize the "three-body closeness of approach" by Λ^2, L_1, and L_2, but this description suffers from its asymmetry. It is apparent from the foregoing that for a short-range interaction potential, the three particles will have the greatest probability of undergoing a true "three-body" collision when the quantum numbers λ and σ (which partially characterize the system) are small.

Although the most appropriate representation for treating 3-body scattering is the symmetric one with wavefunctions denoted[5] by $\langle K\Lambda L\Sigma_t \mid \rho\ {}_i\Theta,\ {}_i\Phi,\ \varphi\rangle$, the coordinates ρ, ${}_i\Theta$, ${}_i\Phi$, and φ are related to the "normalized" center-of-mass system cartesian coordinates ξ_i^1, ξ_i^2 by intractable bilinear forms. Instead we shall employ the asymmetric representation with wavefunctions $\langle K\Lambda LY \mid \rho\chi\phi_+\phi_-\rangle$, or rather its equivalent $\langle K\Lambda L_1L_2 \mid \rho\chi\phi_1\phi_2\rangle$, returning to the set $(K, \Lambda, L\Sigma_t)$ later. The transformation equations connecting (ξ^1, ξ^2) and $(\rho, \chi, \phi_1, \phi_2)$ are

$$\begin{aligned} \xi_1^1 &= \rho \cos\chi \cos\phi_1, & \xi_2^1 &= \rho\cos\chi\sin\phi_1, \\ \xi_1^2 &= \rho\sin\chi\cos\phi_2, & \xi_2^2 &= \rho\sin\chi\sin\phi_2, \end{aligned} \tag{7}$$

in which $\rho^2 = \sum_{i,j=1} (\xi_i^j)^2$.

The Schrödinger equation describing the motion of the three particles can be written in terms of the variables $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ or alternatively in terms of the coordinates $(\rho, \chi, \phi_1, \phi_2)$. The former set yields plane-wave solutions

$$\langle\pi|\xi\rangle = [1/(2\pi)^2] \exp(i\boldsymbol{\pi}\cdot\boldsymbol{\xi}), \tag{8}$$

in which we have chosen our units such that $\hbar = 1$ and the normalization is one particle per unit volume. The scalar product $\boldsymbol{\pi}\cdot\boldsymbol{\xi} = \boldsymbol{\pi}^1\cdot\boldsymbol{\xi}^1 + \boldsymbol{\pi}^2\cdot\boldsymbol{\xi}^2$ can be expressed in terms of the angular coordinates as follows:

$$\boldsymbol{\pi}\cdot\boldsymbol{\xi} = k\rho[\cos\chi\cos\bar{\chi}\cos(\phi_1 - \bar{\Phi}_1) + \sin\chi\sin\bar{\chi}\cos(\phi_2 - \bar{\Phi}_2)], \tag{9}$$

where $k = |\boldsymbol{\pi}|$ and $\rho = |\boldsymbol{\xi}|$; the barred angles fix the direction of $\boldsymbol{\pi}$ and the unbarred angles that of $\boldsymbol{\xi}$. One cannot perform a coordinate rotation to a system in which $\chi = 0$ or $\bar{\chi} = 0$; hence all angles must be retained in the computations.

If we are dealing with a "central" potential, i.e., one in which V is a function of $|\boldsymbol{\xi}|$ only, or one which is "almost" central, the coordinate systems $(\rho\chi\phi_1\phi_2)$ or $\rho\Theta\phi\Phi)$ are appropriate. The solutions of the corresponding Schrödinger equations outside the range of the potential are[5]

$$\langle K\lambda L_1 L_2|\rho\chi\phi_1\phi_2\rangle = (1/k\rho)J_{\lambda+1}(k\rho)\mathcal{J}_{\lambda m_1 m_2}(\chi\phi_1\phi_2), \tag{10a}$$

$$\langle K\lambda L\Sigma_t|\rho\Theta\phi\Phi\rangle = (1/k\rho)J_{\lambda+1}(k\rho)\mathcal{g}_{\lambda m+\sigma}(\Theta\phi\Phi), \tag{10b}$$

where $J_{\lambda+1}(k\rho)$ is the Bessel function of the first kind of order $\lambda + 1$, k is the magnitude of the momentum ($k^2 = 2\mu K$), and

$$\Lambda^2\mathcal{J}_{\lambda m_1 m_2} = \lambda(\lambda + 2)\mathcal{J}_{\lambda m_1 m_2}, \tag{11a}$$

$$L_{1,2}\mathcal{J}_{\lambda m_1 m_2} = m_{1,2}\mathcal{J}_{\lambda m_1 m_2}, \tag{11b}$$

$$\Lambda^2\mathcal{g}_{\lambda m+\sigma} = \lambda(\lambda + 2)\mathcal{g}_{\lambda m+\sigma}, \tag{11c}$$

$$L\mathcal{g}_{\lambda m+\sigma} = m_+\mathcal{g}_{\lambda m+\sigma}, \tag{11d}$$

$$\Sigma_t\mathcal{g}_{\lambda m+\sigma} = \sigma\mathcal{g}_{\lambda m+\sigma}. \tag{11e}$$

$\mathcal{J}_{\lambda m_1 m_2}(\chi\phi_1\phi_2)$ and $\mathcal{g}_{\lambda m+\sigma}(\Theta\phi\Phi)$ are given by

$$\begin{aligned}\mathcal{J}_{\lambda m_1 m_2}(\chi\phi_1\phi_2) &= \{[2(\lambda + 1)]^{\frac{1}{2}}/2\pi\}e^{im_1(\phi_1-\frac{1}{2}\pi)}e^{im_2(\phi_2-\frac{1}{2}\pi)}\\ &\times \{[\tfrac{1}{2}(\lambda - m_1 + m_2)]!\,[\tfrac{1}{2}(\lambda + m_1 - m_2)]!\\ &\times [\tfrac{1}{2}(\lambda - m_1 - m_2)]!\,[\tfrac{1}{2}(\lambda + m_1 + m_2)]!\}^{\frac{1}{2}}\\ &\times \sum_{\kappa=0}^{\frac{1}{2}(\lambda-m_1-m_2)} (-1)^\kappa\{\kappa!\,(m_1 + \kappa)!\\ &\times [\tfrac{1}{2}(\lambda - m_1 + m_2) - \kappa]!\\ &\times [\tfrac{1}{2}(\lambda - m_1 - m_2) - \kappa]!\}^{-1}\\ &\times \sin^{\lambda-m_1-2\kappa}\chi\cos^{m_1+2\kappa}\chi\end{aligned} \tag{12a}$$

$$\mathcal{g}_{\lambda m+\sigma}(\Theta\varphi\Phi) = \{[2(\lambda + 1)]^{\frac{1}{2}}/2\pi\}\,\mathfrak{D}^{\frac{1}{2}\lambda}_{\frac{1}{2}m,-\frac{1}{2}\sigma}(2\phi, \pi - 2\Theta, -2\Phi). \tag{12b}$$

The functions $\mathfrak{D}^\alpha_{\beta\gamma}$ appearing in (12b) are the elements of the Wigner representation of the three-dimensional rotation group.[6]

In the following section we shall compute the free-particle Green's function, after which the elastic and inelastic three-particle scattering amplitude and collision cross section will be treated. In the final section we shall extend our development to the case of identical particles.

[6] E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959).

2. FREE-PARTICLE GREEN'S FUNCTION

In developing the dynamics of a three-body collision it is convenient to employ the integral equation methods of Lippmann and Schwinger.[7,8,9] In this scheme the state vector $|0\rangle$ of the system is related to the incoming state vector $|i\rangle$ by the equation

$$|0\rangle = |i\rangle + G^{(+)}(E_0)V\,|0\rangle, \tag{13}$$

where V is the interaction potential and $G^{(+)}(E_0)$ is the free-particle Green's function given in operator form by

$$G^{(+)}(E_0) = \lim_{\epsilon\to+0}(E_0 - K + i\epsilon)^{-1}. \tag{14}$$

In Eq. (14) K is the free-particle Hamiltonian operator and ϵ is an adiabatic switching parameter introduced in order to avoid transients in the time dependence of the state function during the scattering process; it has the significance of feeding in the incident wave over a period of time rather than releasing it suddenly.[8] The positive sign adjacent to ϵ in (14) ensures outgoing scattered waves. In position coordinate representation we have

$$\langle\xi|0\rangle = \langle\xi|i\rangle + \int G^{(+)}_{E_0}(\xi, \xi')V(\xi')\langle\xi'|0\rangle\,d\xi', \tag{15}$$

where

$$\begin{aligned}G^{(+)}_{E_0}(\xi, \xi') &= \int \langle\xi|\boldsymbol{\pi}\rangle\langle\boldsymbol{\pi}|G^{(+)}(E_0)|\boldsymbol{\pi}\rangle\langle\boldsymbol{\pi}|\xi'\rangle\,d\boldsymbol{\pi}\\ &= \lim_{\epsilon\to+0}\frac{2\mu}{(2\pi)^4}\int\frac{\exp[i\boldsymbol{\pi}\cdot(\xi - \xi')]}{k_0^2 - k^2 + 2i\mu\epsilon}\,d\boldsymbol{\pi};\quad k = |\boldsymbol{\pi}|,\end{aligned} \tag{16}$$

using the linear momentum representation of the operator $G^{(+)}_E$; the symbol μ represents the reduced mass and the integral is taken over all momentum space $\boldsymbol{\pi}$. The scalar products $\boldsymbol{\pi}\cdot\xi$ are now expressed in terms of the hyperpolar coordinates (χ, ϕ_1, ϕ_2) as in Eq. (9). After carrying out the integration over the azimuthal angles ϕ_1, ϕ_2, we obtain

$$\begin{aligned}G^{(+)}_{k_0}(\xi, \xi') &= \lim_{\epsilon\to+0}\frac{2\mu}{(2\pi)^4}\\ &\times \int_0^\infty\int_0^{\frac{1}{2}\pi}\frac{J_0(kr_1\cos\chi)J_0(kr_2\sin\chi)}{k^2 - k^2 + 2i\mu\epsilon}\\ &\times \tfrac{1}{2}\sin 2\chi\,d\chi k^3\,dk,\end{aligned} \tag{17}$$

[7] B. A. Lippmann and J. Schwinger, Phys. Rev. 79, 469 (1950).
[8] M. Gell-Mann and M. L. Goldberger, Phys. Rev. 91, 398 (1953).
[9] G. Gioumousis and C. F. Curtis, J. Chem. Phys. 29, 996 (1958).

where

$$r_1 = [\xi^2 \cos^2 \bar{\chi} + \xi'^2 \cos^2 \bar{\bar{\chi}} - 2\xi\xi' \cos \bar{\chi} \cos \bar{\bar{\chi}} \cos (\bar{\phi}_1 - \bar{\bar{\phi}}_1)]^{\frac{1}{2}},$$

$$r_2 = [\xi^2 \sin^2 \bar{\chi} + \xi'^2 \sin^2 \bar{\bar{\chi}} - 2\xi\xi' \sin \bar{\chi} \sin \bar{\bar{\chi}} \cos (\bar{\phi}_2 - \bar{\bar{\phi}}_2)]^{\frac{1}{2}},$$

and the directions of ξ and ξ' are specified by the sets of angles $(\bar{\chi}\bar{\phi}_1\bar{\phi}_2)$ and $(\bar{\bar{\chi}}\bar{\bar{\phi}}_1\bar{\bar{\phi}}_2)$, respectively.

Using a theorem due to Sonine,[10] one can immediately integrate over χ, obtaining the result

$$G_{k_0}^{(+)}(\xi, \xi') = \lim_{\epsilon \to +0} \frac{2\mu}{(2\pi)^2} \frac{1}{r} \int_0^\infty \frac{J_1(kr)}{k_0^2 - k^2 + 2i\mu\epsilon} k^2\, dk$$

$$= \lim_{\epsilon \to +0} \frac{2\mu}{(2\pi)^2} \frac{1}{r} \frac{\partial}{\partial r} \left[\int_0^\infty \frac{kJ_0(kr)}{k_0^2 - k^2 + 2i\mu\epsilon} dk \right], \quad (18)$$

in which $r = (r_1^2 + r_2^2)^{\frac{1}{2}}$. The integral in (18) which we shall call I can be evaluated by constructing a contour of integration in the complex plane such that outgoing waves are guaranteed. To do this, one first expresses $J_0(kr)$ as a sum of Hankel functions of the first and second kinds;

$$J_0(\chi) = \tfrac{1}{2}[H_0^{(1)}(\chi) + H_0^{(2)}(\chi)], \quad (19)$$

thus recasting the integral I in the form

$$I = \lim_{\epsilon \to +0} \frac{1}{2} \int_0^\infty \frac{H_0^{(1)}(kr)k\, dk}{k_0^2 - k^2 + 2i\mu\epsilon} + \lim_{\epsilon \to +0} \frac{1}{2} \int_0^\infty \frac{H_0^{(2)}(kr)k\, dk}{k_0^2 - k^2 + 2i\mu\epsilon}. \quad (20)$$

The first integral I_1 in (20) is evaluated by means of Cauchy's integral formula and the appropriate contour of integration shown in Fig. 1(a).

$$\lim_{\epsilon \to +0} \left[\frac{1}{2} \int_0^\infty \frac{H_0^{(1)}(kr)}{k_0^2 - k^2 + 2i\mu\epsilon} k\, dk + \frac{1}{2} \int_C \frac{H_0^{(1)}(kr)}{k_0^2 - k^2 + 2i\mu\epsilon} k\, dk + \frac{1}{2} \int_{i\infty}^0 \frac{H_0^{(1)}(kr)}{k_0^2 - k^2 + 2i\mu\epsilon} k\, dk \right]$$

$$= 2\pi i \operatorname{Res}(k_0). \quad (21)$$

The second integral in (21) vanishes because $H_0^{(1)}(Z)$ approaches zero in the limit $|Z|$ approaches infinity. Using the contour of integration shown in Fig. 1(b), the second integral I_2 in (20) is computed in a similar manner. Substituting the solutions of I_1 and I_2 into

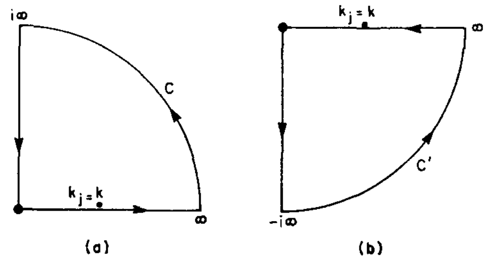

FIG. 1. Contours of integration for integrals I_1 and I_2 [Eq. (20)].

(18), we obtain

$$G_{k_0}^{(+)}(\xi, \xi') = \frac{i\mu}{4\pi} \frac{k}{r} H_1^{(1)}(kr), \quad (22)$$

or, asymptotically for large ξ,

$$G_{k_0}^{(+)}(\xi, \xi') = e^{-i\pi/4} \frac{\mu k^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} \frac{e^{ik\xi}}{\xi^{\frac{3}{2}}} e^{-ik\xi'\alpha}, \quad (23)$$

where

$$\alpha = \cos\chi \cos\chi' \cos(\phi_1 - \phi_1') + \sin\chi \sin\chi' \cos(\phi_2 - \phi_2'). \quad (24)$$

In addition to expressing Eq. (15) in the linear momentum representation we can also cast it in terms of generalized angular momentum eigenfunctions

$\langle \xi | K\lambda m_1 m_2 \rangle$ or $\langle \xi | K\lambda m_+ \sigma \rangle$;

$$G_{k_0}^{(+)}(\xi, \xi') = \sum_{\lambda m_1 m_2} \langle \xi | K\lambda m_1 m_2 \rangle \times \langle K\lambda m_1 m_2 | G_{k_j}^{(+)} | K\lambda m_1 m_2 \rangle \langle K\lambda m_1 m_2 | \xi' \rangle$$

$$= \lim_{\epsilon \to +0} 2\mu \sum_{\lambda m_1 m_2} \mathcal{J}_{\lambda m_1 m_2}(\hat{\xi}) \mathcal{J}^*_{\lambda m_1 m_2}(\hat{\xi}') \frac{1}{\xi\xi'} \times \int_0^\infty \frac{J_{\lambda+1}(k\xi) J_{\lambda+1}(k\xi')}{k_0^2 - k^2 + 2i\mu\epsilon} k\, dk$$

$$= \pi\mu i \sum_{\lambda m_1 m_2} \mathcal{J}_{\lambda m_1 m_2}(\hat{\xi}) \mathcal{J}^*_{\lambda m_1 m_2}(\hat{\xi}') \times \frac{1}{\xi\xi'} H_{\lambda+1}^{(1)}(k_0\xi) J_{\lambda+1}(k_0\xi'); \qquad \xi > \xi', \quad (25)$$

and

$$G_{k_0}^{(+)}(\xi, \xi') = \sum_{\lambda m_+ \sigma} \langle \xi | K\lambda m_+ \sigma \rangle \times \langle K\lambda m_+ \sigma | G_{k_j}^{(+)} | K\lambda m_+ \sigma \rangle \langle K\lambda m_+ \sigma | \xi' \rangle$$

$$= \pi\mu i \sum_{\lambda m_+ \sigma} \mathcal{J}_{\lambda m_+ \sigma}(\hat{\xi}) \mathcal{J}^*_{\lambda m_+ \sigma}(\hat{\xi}') \times \frac{1}{\xi\xi'} H_{\lambda+1}^{(1)}(k_0\xi) J_{\lambda+1}(k_0\xi'); \qquad \xi > \xi'. \quad (26)$$

It is interesting to note that the Green's function of N free particles moving in a plane can be derived

[10] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, London, 1952), 2nd Ed., p. 376.

with the aid of a suitable generalization of the above method. The asymptotic form

$$G_k^{(+)}(\xi, \xi')_N = \frac{e^{-i(\frac{1}{4}+\frac{1}{2}N)\pi}}{(2\pi)^{N-\frac{1}{2}}} \frac{\mu k^{N-5/2}}{|\xi - \xi'|^{N-\frac{1}{2}}} \exp(ik|\xi - \xi'|) \tag{27}$$

is derived in Appendix A. The vectors ξ and π are generalizations of Eq. (2) to N bodies:

$$\xi = \sum_{i=1}^{N-1} \xi^i,$$

$$\pi = \sum_{i=1}^{N-1} \pi^i.$$

3. THREE-PARTICLE ELASTIC SCATTERING

Having derived the Green's function we are now in a position to investigate three-particle scattering dynamics.

Asymptotically the wavefunction of the system takes the form

$$\Psi_{\pi_i}(\xi) \sim \phi_{\mathbf{k}_i}(\xi) + e^{-i\pi/4} \frac{k^{\frac{1}{2}}}{\xi^{\frac{5}{2}}} e^{ik\xi} f(\hat{\pi}_i, \hat{\pi}_0), \tag{28}$$

where $f(\hat{\pi}_i, \hat{\pi}_0)$ is the three-body scattering amplitude given by

$$f(\hat{\pi}_i, \hat{\pi}_0) = \frac{\mu}{(2\pi)^{\frac{5}{2}}} \int_0^\infty \int_0^{\frac{1}{2}\pi} \int_0^{2\pi} \int_0^{2\pi} e^{-ik\xi'\alpha} V(\xi'\chi'\phi_1'\phi_2'),$$
$$\times \Psi_{\pi_i}(\xi'\chi'\phi_1'\phi_2') \tfrac{1}{2} \sin 2\chi' \, d\chi' \, d\phi_1' \, d\phi_2' \xi'^3 \, d\xi'. \tag{29}$$

The symbol α has the same significance as in (24) and $\hat{\pi}_i$ and $\hat{\pi}_0$ represent the directions of the incoming and scattered three-particle momenta, respectively. Alternatively, the scattering amplitude can be expanded, with the aid of (25), in a series of generalized angular momentum eigenfunctions

$$f(\hat{\pi}_i, \hat{\pi}_0) = (2\pi)^{\frac{1}{2}} \frac{\mu}{k} \sum_{\lambda m_1 m_2} (-1)^\lambda \mathcal{G}_{\lambda m_1 m_2}(\hat{\pi}_i)$$
$$\times \int \frac{J_{\lambda+1}(k\xi')}{\xi'} \mathcal{G}^*_{\lambda m_1 m_2}(\hat{\xi}') V(\xi') \Psi_{\pi_i}(\xi') \, d\xi'. \tag{30}$$

For many purposes it is more useful to express the scattering amplitude in the generalized angular momentum representations $(\lambda m_1 m_2)$ or $(\lambda m_+ \sigma)$ rather than as an explicit function of the coordinates $(\chi\phi_1\phi_2)$ or $(\Theta\phi\Phi)$. The transformation coefficients are just the generalized angular momentum eigenfunctions

$$f(\chi\phi_1\phi_2; \chi'\phi_1'\phi_2') = \sum_{\substack{\lambda m_1 m_2 \\ \lambda' m_1' m_2'}} \mathcal{G}^*_{\lambda m_1 m_2}(\chi\phi_1\phi_2) f^{\lambda\lambda'}_{m_1 m_1'; m_2 m_2'} \mathcal{G}_{\lambda' m_1' m_2'}(\chi'\phi_1'\phi_2'), \tag{31}$$

$$f(\Theta\phi\Phi; \Theta'\phi'\Phi') = \sum_{\substack{\lambda m + \sigma \\ \lambda' m' \sigma'}} \mathcal{G}^*_{\lambda m_+ \sigma}(\Theta\phi\Phi) f^{\lambda\lambda'}_{m_+ m_+'; \sigma\sigma'} \mathcal{G}_{\lambda' m_+' \sigma'}(\Theta'\phi'\Phi'). \tag{32}$$

Inverting the expansions (31) and (32) and employing (30), we obtain, for $f^{\lambda\lambda'}_{m_1 m_1'; m_2 m_2'}$ and $f^{\lambda\lambda'}_{m_+ m_+'; \sigma\sigma'}$,

$$f^{\lambda\lambda'}_{m_1 m_1'; m_2 m_2'} = (2\pi)^{\frac{1}{2}} \frac{\mu}{k} (-1)^\lambda \int \frac{J_{\lambda+1}(k\xi')}{\xi'} \mathcal{G}^*_{\lambda m_1 m_2}$$
$$\times (\hat{\xi}') V(\xi') \Psi_{\lambda' m_1' m_2'}(\xi') \, d\xi'$$
$$= f^{\lambda\lambda'}_{m_+ m_+'; m_- m_-'} e^{i\beta}, \tag{33}$$

$$f^{\lambda\lambda'}_{m_+ m_+; \sigma\sigma'} = (2\pi)^{\frac{1}{2}} \frac{\mu}{k} (-1)^\lambda$$
$$\times \int \frac{J_{\lambda+1}(k\xi')}{\xi'} \mathcal{G}^*_{\lambda m_+ \sigma}(\hat{\xi}') V(\xi') \Psi_{\lambda' m' \sigma'}(\xi') \, d\xi', \tag{34}$$

where β is merely a phase angle. The scattering amplitude is easily transformed from one generalized angular momentum representation to the other by means of the unitary transformation

$$f^{\lambda\lambda'}_{m_+ m_+' \sigma\sigma'} = \sum_{m_- m_-'} b_\lambda(m_-, \sigma)$$
$$\times f^{\lambda\lambda'}_{m_+ m_+'; m_- m_-'} b_{\lambda'}(m_-', \sigma'). \tag{35}$$

A variational method for computing the scattering amplitude corresponding to the two-particle collision process has been developed by Schwinger.[11] This procedure can be suitably adapted to the three-particle case by expressing the scattering amplitude as

$$f(\hat{\pi}_0, \hat{\pi}_i) = \frac{\left[\int \psi^*(\xi) U(\xi) \exp(i\pi_i \cdot \xi) \, d\xi\right]\left[\int \exp(-i\pi_0 \cdot \xi') U(\xi') \psi(\xi') \, d\xi'\right]}{\left[\psi^*(\xi) U(\xi) \psi(\xi) \, d\xi - \frac{k^{\frac{1}{2}} e^{-i\pi/4}}{(2\pi)^{\frac{5}{2}}} \iint \psi^*(\xi) U(\xi) \frac{e^{ikR}}{R^{\frac{5}{2}}} U(\xi') \psi(\xi') \, d\xi' \, d\xi\right]}, \tag{36}$$

where $R = |\xi - \xi'|$ and $U(\xi) = \mu V(\xi)$. Variation of f with respect to ψ^* then yields Eq. (29). The utility of this principle lies in computing wavefunctions ψ which make the scattering amplitude on extremum, and therefore presumably give a best fit to it. In practice this is carried out by writing the wave-

[11] D. R. Bates, *Quantum Theory I. Elements* (Academic Press Inc., New York, 1961), p. 350.

function of the system as a function of various parameters, e.g.,

$$\psi = \sum_i \alpha_i \phi_i. \tag{37}$$

Variation of f with respect to the expansion parameters α_i results in the set of simultaneous equations

$$\partial f/\partial \alpha_i = 0, \tag{38}$$

which are then solved for α_i. This procedure, of course, yields only an upper or lower bound to the magnitude of the scattering amplitude depending upon whether the potential is repulsive or attractive; the "goodness" of the approximation depends upon an adroit choice of the basis (ϕ_i). It is hoped that this approach can be exploited in the eventual treatment of actual three-body problems.

In two-particle scattering, the differential scattering cross section $\sigma(\theta, \varphi)$ is related to the scattering amplitude by the equality

$$\sigma(\theta, \varphi) = |f(\theta, \varphi)|^2, \tag{39}$$

where (θ, φ) fix the orientation of the outgoing with respect to the incoming momentum. Derivation of the three-particle analogue of (39) can be approached in two ways which, of course, must yield the same result. The first one is based on the elementary definition of cross section

$$\sigma(\boldsymbol{\pi}_0, \boldsymbol{\pi}_i) = [|\mathbf{j}_{\text{out}}|/|\mathbf{j}_{\text{in}}|]\xi^3\, d\Omega, \tag{40}$$

where $\mathbf{j}_{\text{in}}$ and $\mathbf{j}_{\text{out}}$ are the probability current densities of the incoming and outgoing particles, respectively, and $\xi^3\, d\Omega$ is an element of "surface area" of a large hypersphere the center of which is coincident with the center of scattering. The probability current densities are given by

$$\mathbf{j}_{\text{out}} = (1/2i\mu)[\psi_{\text{sc}}^* \nabla \psi_{\text{sc}} - (\nabla \psi_{\text{sc}}^*)\psi_{\text{sc}}], \tag{41a}$$

$$\mathbf{j}_{\text{in}} = (1/2i\mu)[\phi_{\text{i}}^* \nabla \phi_{\text{i}} - (\nabla \phi_{\text{i}}^*)\phi_{\text{i}}], \tag{41b}$$

where ψ_{sc} and ϕ_{i} are the scattered and incident wavefunctions, respectively and μ is the reduced mass. The normalized wavefunctions ϕ_{i} and ψ_{sc} are

$$\phi_{\text{i}}(\boldsymbol{\xi}) = [1/(2\pi)^2]\exp(i\boldsymbol{\pi}\cdot\boldsymbol{\xi}), \tag{42a}$$

$$\psi_{\text{sc}}(\boldsymbol{\xi}) = (k^{\frac{1}{2}}/\xi^{\frac{3}{2}})e^{-i\pi/4}e^{ik\xi}f(\hat{\pi}_0, \hat{\pi}_i), \tag{42b}$$

from which we obtain, with the aid of (41) and (40),

$$\sigma(\boldsymbol{\pi}_0, \boldsymbol{\pi}_i) = k\, |f(\hat{\pi}_0, \hat{\pi}_i)|^2. \tag{43}$$

The cross section σ has the dimensions of length cubed since $f(\hat{\pi}_0, \hat{\pi}_i)$ can be shown to have dimensions of length squared.

We make the second approach via the appropriate matrix element of the transition rate amplitude R_{π_0, π_i} corresponding to the collision process

$$\sigma(\boldsymbol{\pi}_0, \boldsymbol{\pi}_i) = (2\pi/V_0)\rho(E)\, |R_{\pi_0, \pi_i}|^2, \tag{44}$$

where V_0 is the velocity of the incoming particles given by $V_0 = k/\mu$, and $\rho(E)$ is the number of final states per unit energy:

$$\rho(E) = \frac{k^3}{(2\pi)^4}\frac{dk}{dE}\, d\Omega = \frac{\mu k^2}{(2\pi)^4}\, d\Omega. \tag{45}$$

The matrix element in question is $\langle 0\, |V|\, i\rangle$ which can be recast in the more useful form[9]

$$R_{\pi_0, \pi_i} = \lim_{\epsilon\to+0} -i\epsilon\langle\boldsymbol{\pi}_0|\boldsymbol{\pi}_i\rangle, \tag{46}$$

or, explicitly in terms of (42),

$$R_{\pi_0, \pi_i} = \lim_{\epsilon\to+0}(-i\epsilon)\int_0^\infty\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{2\pi}\exp(-i\boldsymbol{\pi}_0\cdot\boldsymbol{\xi})$$
$$\times\left[\exp(i\boldsymbol{\pi}_i\cdot\boldsymbol{\xi}) + \frac{k^{\frac{1}{2}}}{\xi^{\frac{3}{2}}}e^{ik\xi}e^{-i\pi/4}e^{-\epsilon\mu\xi/k}f(\hat{\pi}_0, \hat{\pi}_i)\right]\xi^3\, d\xi\, d\Omega, \tag{47}$$

in which the factor $e^{-\epsilon\mu\xi/k}$ is introduced because of the requirement that we replace the energy E by $E + i\epsilon$ when we move into the first quadrant of the complex plane. The term containing $\exp[i(\boldsymbol{\pi}_i - \boldsymbol{\pi}_0)\cdot\boldsymbol{\xi}]$ contributes a delta function which is of no interest since it indicates *no scattering*. In order to carry out the second integration, one must expand the plane wave (momentum eigenfunction) $\exp(-i\boldsymbol{\pi}_0\cdot\boldsymbol{\xi})$ in a series of generalized angular momentum eigenfunctions $\mathcal{G}_{\lambda m_1 m_2}$ (see Appendix B). Making this substitution in (39), and changing the lower limit of the ξ integration to some large but finite value r, we obtain

$$R_{\pi_0, \pi_i} = \lim_{\epsilon\to+0}(-i\epsilon)\sum_{\lambda m_1 m_2}(2\pi)^2(-i)^\lambda$$
$$\times\int_0^\infty\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{J_{\lambda+1}(k\xi)e^{ik\xi}}{k^{\frac{1}{2}}\xi^{5/2}}$$
$$\times e^{-\epsilon\mu\xi/k}e^{-i\pi/4}\mathcal{G}_{\lambda m_1 m_2}(\chi''\phi_1''\phi_2'')$$
$$\times \mathcal{G}^*_{\lambda m_1 m_2}(\chi'\phi_1'\phi_2')f(\chi\phi_1\phi_2;\chi'\phi_1\phi_2)$$
$$\times \xi^3\, d\xi \sin\chi' \cos\chi'\, d\chi'\, d\phi_1'\phi_2'. \tag{48}$$

The artifice of assuming a large finite value for the lower limit of the integral over ξ is quite legitimate; the integral $\int_0^\infty$ can be split into a sum $\int_r^\infty + \int_0^r$ wherein the second integral $\int_0^r$ is finite and therefore yields no contribution to R_{π_0, π_i} in the limit ϵ approaches $+0$. Since the value of ξ is large in the region of integration, the Bessel function $J_{\lambda+1}(k\xi)$ can be replaced by its asymptotic form. With the aid of (31) we obtain

$$R_{\pi_0, \pi_i} = [(2\pi)^{\frac{3}{2}}/\mu]f(\hat{\pi}_0, \hat{\pi}_i). \tag{49}$$

Finally one again arrives at the relation between the scattering amplitude and cross section when the form of R_{π_0,π_i} given by (49) is substituted into (44).

$$\sigma(\pi_0\pi_i) = k\,|f(\hat{\pi}_0, \hat{\pi}_i)|^2, \tag{43}$$

wherein $f(\hat{\pi}_0, \hat{\pi}_i)$ can be represented in $(\chi\phi_1\phi_2)$, $(\chi\phi_-\phi_+)$ or in $({}_i\Theta, {}_i\Phi, \varphi)$ coordinates.

4. PARTIAL-WAVE ANALYSIS

One of the most useful methods of analyzing low-energy two-body scattering is the expansion of the scattering amplitude in "partial waves" each of which is associated with a definite orbital angular momentum. The simplest case is that of a "central" short-range potential, i.e., $V = V(|\xi|)$. Although of very great importance in two-particle scattering, this appears at first glance to be a very unphysical interaction when one extends it to the three-particle case. Nevertheless, consideration of the connection between $|\xi|$ and the moment of inertia of the three particles about an axis through the center of mass and perpendicular to the plane of the particles, i.e., $I = \mu\xi^2$ where μ is the reduced mass and I is the moment of inertia, shows that it is reasonable, at least to first approximation. If the particles must simultaneously be within a certain distance of each other for the potential to act, the moment of inertia I must be less than some value I_0. A "square well" potential would then have the form

$$\begin{aligned} V(|\xi|) &= V_0; \qquad (I_0/\mu) < \xi_0^2 \\ &= O\ (I_0/\mu) > \xi_0^2. \end{aligned} \tag{50}$$

If (50) does represent the interaction potential, the solution to the wave equation can be separated into radial and angular parts; the general solution is of the form

$$\Psi_{\pi_i}(\xi) = \sum_{\lambda m_1 m_2} A_{\lambda m_1 m_2} \frac{R_\lambda(k\xi)}{k\xi}\, \mathcal{J}^*_{\lambda m_1 m_2}(\hat{\pi}_i)\mathcal{J}_{\lambda m_1 m_2}(\hat{\xi}), \tag{51}$$

where $R_\lambda(k\xi)$ approaches

$$(1/k\xi)^{\frac{1}{2}} \sin\left[k\xi - \tfrac{1}{4}(2\lambda + 1)\pi + \eta_\lambda\right]$$

asymptotically; the η_λ are the *three-body* λ-dependent scattering phase shifts. Now the wavefunction of the system, $\Psi_{\pi_0}(\xi)$, can also be expressed as a function of the scattering amplitude (28). In addition, the asymptotic form of (51) can be separated into a part containing the common factor $e^{ik\xi}$ and another part containing the common factor $e^{-ik\xi}$: After the expansion of $\phi_i(\xi)$ in generalized angular momentum eigenfunctions (see Appendix B), we can equate the coefficients of $e^{ik\xi}$ and $e^{-ik\xi}$ appearing in the two forms of $\Psi_{\pi_i}(\xi)$ and solve for $A_{\lambda m_1 m_2}$:

$$A_{\lambda m_1 m_2} = (2\pi)^2 (i)^\lambda e^{i\eta_\lambda}, \tag{52}$$

and

$$f(\hat{\pi}_0, \hat{\pi}_i) = \frac{(2\pi)^{\frac{3}{2}}}{ik^2} \sum_{\lambda m_1 m_2} (e^{2i\eta_\lambda} - 1)\mathcal{J}^*_{\lambda m_1 m_2}(\hat{\pi}_0)\mathcal{J}_{\lambda m_1 m_2}(\hat{\pi}_i). \tag{53}$$

By comparing (53) with (31) and (33) it becomes evident that

$$\begin{aligned} f^{\lambda\lambda'}_{m_1 m_1',\, m_2 m_2'} &= \frac{(2\pi)^{\frac{3}{2}}}{ik^2}(e^{2i\eta_\lambda} - 1)\,\delta_{\lambda\lambda'}\,\delta_{m_1 m_1'}\,\delta_{m_2 m_2'} \\ &= f^\lambda = f^{\lambda\lambda'}_{m_+ m_+',\, \sigma\sigma'}. \end{aligned} \tag{54}$$

More complicated cases involving "noncentral" potentials will result in more complicated expressions for $f^{\lambda\lambda'}_{m_1 m_1', m_2 m_2'}$ or alternatively $f^{\lambda\lambda'}_{m_+ m_+', \sigma\sigma'}$, which have off-diagonal terms and involve phase shifts labeled by m_1 and $m_2 (m_+$ and $\sigma)$. A general treatment of such potential functions, an example of which is

$$\begin{aligned} V(\xi) &= V_0(|\xi|)(1 + \alpha\xi^1\cdot\xi^2); \\ &\qquad (\xi)^2 + \alpha\xi^1\cdot\xi^2 < (\xi_0)^2 \\ &= 0; \qquad (\xi)^2 + \alpha\xi^1\cdot\xi^2 > (\xi_0)^2, \end{aligned} \tag{55}$$

will be deferred for a future paper.

INELASTIC SCATTERING

Inelastic scattering is possible if one or more of the particles has internal structure or two of the particles can combine. Typical examples of the latter are three-body attachment of electrons to neutral atoms, three-body electron–ion recombination, and formation of diatomic molecules by means of three-body atomic collisions. The case in which the number of free-particles is conserved will be considered first.

If the three particles all have internal structure, the free-particle wavefunctions are of the form

$$\phi_i = [1/(2\pi)^2]R_m(\rho)S_n(\sigma)T_p(\tau) \times \exp\left[i(\pi_\alpha^1\cdot\xi^1 + \pi_\alpha^2\cdot\xi^2)\right], \tag{56}$$

where $R_m(\rho)$, $S_n(\sigma)$, and $T_p(\tau)$ represent the internal structures of the particles. The Green's function (14) of the three-body interaction then takes the form

$$G_E^{(+)}(\xi, \xi') = \lim_{\epsilon\to+0} \frac{1}{(2\pi)^4} \sum_{\mu\nu\lambda} \int \frac{\exp\left[i\pi_\alpha\cdot(\xi - \xi')\right]R_\mu(\rho)S_\nu(\sigma)T_\lambda(\tau)R^*_\mu(\rho')S^*_\nu(\sigma')T^*_\lambda(\tau')}{E - E_\alpha - E_\lambda - E_\nu - E_\mu + i\epsilon}\, d\pi_\alpha. \tag{57}$$

If we now make the substitution

$$(1/2\mu)k_{\mu\nu\lambda}^2 = E - E_\lambda - E_\nu - E_\mu, \qquad (58)$$

and carry out the integrations indicated in (57) using the methods of Sec. 2, we obtain

$$G_E^{(+)}(\xi\rho\sigma\tau;\ \xi'\rho'\sigma'\tau') = \frac{\sum_{\mu\nu\lambda} i\mu k_{\mu\nu\lambda} H_1^{(1)}(k_{\mu\nu\lambda}\,|\xi - \xi'|)}{4\pi\,|\xi - \xi'|}$$
$$\times R_\mu(\rho)S_\nu(\sigma)T_\lambda(\tau)R_\mu^*(\rho')S_\nu^*(\sigma')T_\lambda^*(\tau') \qquad (59)$$

for the Green's function, and thus,

$$\psi(\xi, \rho, \sigma, \tau) = \varphi(\xi, \rho, \sigma, \tau) + \int G_E^{(+)}(\xi, \rho\sigma\tau, \xi'\rho'\sigma'\tau')V(\xi', \rho'\sigma'\tau')$$
$$\times \psi(\xi', \rho', \sigma', \tau')\, d\xi'\, d\rho'\, d\sigma'\, d\tau' \qquad (60)$$

for the wavefunction of the system. As in Eq. (56), the wavefunction $\psi(\xi, \rho, \sigma, \tau)$ is separable and can in general be written

$$\psi(\xi, \rho, \sigma, \tau) = \sum_{\mu\nu\lambda} \psi_{\mu\nu\lambda}(\xi)R_\mu(\rho)S_\nu(\sigma)T_\lambda(\tau), \qquad (61)$$

which allows the "external" wavefunction to be expressed as

$$\psi_{\mu\nu\lambda}(\xi) = \phi_{\mu''\nu''\lambda''}(\xi)\,\delta_{\mu\mu'',\nu\nu'',\lambda\lambda''} + \sum_{\mu'\nu'\lambda''} \int G_{\mu\mu\lambda}^{(+)}(\xi, \xi')\langle\mu\nu\lambda\,|V(\xi')|\,\mu'\nu'\lambda'\rangle$$
$$\times \psi_{\mu'\nu'\lambda'}(\xi')\, d\xi', \qquad (62)$$

where

$$\langle\mu\nu\lambda\,|V(\xi')|\,\mu'\nu'\lambda'\rangle = \int R_\mu(\rho')S_\nu(\sigma')T_\lambda(\tau')V(\xi', \rho', \sigma', \tau')$$
$$\times R_{\mu'}^*(\rho')S_{\nu'}^*(\sigma')T_{\lambda'}^*(\tau')\, d\rho'\, d\sigma'\, d\tau'. \qquad (63)$$

Taking the asymptotic form of $G_E^{(+)}$ [Eq. (23)], we obtain for (63)

$$\psi_{\pi^i\mu\nu\lambda}(\xi) = \phi_{\mu''\nu''\lambda''}(\xi)\,\delta_{\mu\mu'',\nu\nu'',\lambda\lambda''} + [e^{-i\pi/4}k_{\mu\nu\lambda}^{\frac12}e^{ik_{\mu\nu\lambda}}/(2\pi)^{\frac12}\xi^{\frac12}]f(\hat\pi_{\mu\nu\lambda}^i, \hat\pi_{\mu''\nu''\lambda''}^0). \qquad (64)$$

The scattering amplitude is then given by

$$f(\hat\pi_{\mu\nu\lambda}^i, \hat\pi_{\mu'\nu'\lambda'}^0) = \frac{\mu}{(2\pi)^{\frac12}} \int \exp(-ik_{\mu\nu\lambda}\xi'\alpha)$$
$$\times \langle\mu\nu\lambda\,|V(\xi')|\,\mu'\nu'\lambda'\rangle\psi_{\pi^i,\mu'\nu'\lambda'}(\xi')\, d\xi'. \qquad (65)$$

If two of the particles, which we shall assume to be structureless, are bound together in the final state, we have a situation in which $|\pi_2|$ is imaginary, or equivalently

$$\chi = \tfrac12\pi + i\alpha,$$

where α is real. The Green's function of system then takes the form

$$G_{E'}(\xi, \xi') = \lim_{\epsilon\to+0} \frac{2\mu}{(2\pi)^2} \times \sum_n \int \frac{\exp[i\pi_2\cdot(\xi_2 - \xi_2')]}{k'^2 - k^2 + 2i\mu\epsilon}\phi_n(\xi_1)\phi_n^*(\xi_1')\, d\pi_2$$
$$= \lim_{\epsilon\to+0} \frac{2\mu}{(2\pi)^2} \sum_n \phi_n(\xi_1)\phi_n^*(\xi_1') \times \int \frac{\exp(ik\cosh\alpha_n\,|\xi_2 - \xi_2'|\cos\phi_2)}{k^2 - k^2 + 2i\mu\epsilon}k\, dk\, d\phi_2$$
$$= [2\mu/(2\pi)^2] \sum_n \phi_n(\xi_1)\phi_n^*(\xi_1')\tfrac12(\pi i) \times H_0^{(1)}(k\cosh\alpha_n\,|\xi_2 - \xi_2'|), \qquad (66)$$

where $\phi_n(\xi_1)$ are the energy eigenfunctions of the bound (12) pair. Because the energy spectrum of this pair is discrete, the "angles" α_n can assume certain allowed values only, which are related to the internal energy of the pair E_{12} by the equality

$$E_{12}^n = -(1/2\mu)k^2\sinh^2\alpha_n. \qquad (67)$$

In the asymptotic region, $G_{E',n}^{(+)}$ takes the form

$$G_{E',n}^{(+)}(\xi, \xi') = \frac{\mu e^{i\pi/4}}{(2\pi)^{\frac12}}\phi_n(\xi_1)\phi_n^*(\xi_1') \times \frac{\exp(ik_n\,|\xi_2 - \xi_2'|)}{k_n^{\frac12}\,|\xi_2 - \xi_2'|^{\frac12}}, \qquad (68)$$

where $k_n \equiv k\cosh\alpha_n$. Using Eqs. (15) and (68) one can immediately write down the wavefunction of the system:

$$\psi_{\pi^i}(\xi) = \sum_n \frac{e^{i\pi/4}e^{ik_n}}{(k_n\xi_2)^{\frac12}}\phi_n(\xi_1)f(\hat\pi_n^0, \hat\pi^i), \qquad (69)$$

where

$$f(\hat\pi_n^0, \hat\pi^i) = \frac{\mu}{(2\pi)^{\frac12}} \times \int \exp(-i\pi_{2n}\cdot\xi_2)\phi_n(\xi_1)V(\xi)\psi_{\pi^i}(\xi)\, d\xi \qquad (70)$$

is the three-body scattering amplitude corresponding to the rearrangement process. This treatment can be readily generalized to the case of particles with internal structure.

Computation of the inelastic cross section in terms of the scattering amplitude (65), by the methods of Sec. 4, yields

$$\sigma_{\pi_n^0,\pi_{n'}^i}(n' \to n) = (k_n^2/k_{n'})\,|f(\hat\pi_n^0, \hat\pi_{n'}^i)|^2, \qquad (71)$$

where $k_{n'}$ represents the momentum of the incident particles, and n' and n represent the quantum numbers of the internal states of the three incident particles before and after scattering, respectively.

The cross section for scattering into a given exit channel can also be expressed in terms of the three-particle scattering matrix which connects that exit channel with the entrance channel. Normalizing the wavefunction of the system to unit flux in the entrance channel we can write the hyperradial part as[12]

$$R_{\gamma\gamma'}(\xi) = \left(\frac{2\pi}{k_\gamma \xi}\right)^{\frac{5}{2}} \frac{1}{(V_{\gamma'})^{\frac{1}{2}}} \times \left\{\delta_{\gamma\gamma'} \exp\left[-i\left(k_{\gamma'}\xi - \frac{2\lambda+1}{4}\pi\right)\right] - S_{\gamma\gamma'} \exp\left[i\left(k_{\gamma'}\xi - \frac{2\lambda+1}{4}\pi\right)\right]\right\}, \quad (72)$$

the angular part as

$$F_{\gamma'}(\bar{\chi}\bar{\phi}_1\bar{\phi}_2; \chi\phi_1\phi_2) = \mathcal{J}^*_{\lambda_{\gamma'} m_{1\gamma'} m_{2\gamma'}}(\bar{\chi}\bar{\phi}_1\bar{\phi}_2)\mathcal{J}_{\lambda_{\gamma'} m_{1\gamma'} m_{2\gamma'}}(\chi\phi_1\phi_2)$$

in the $(\lambda m_1 m_2)$ representation, and the internal part as $I(\rho, \sigma, \tau)$, where ρ, σ, and τ are the internal coordinates of the three particles. The total flux in the outgoing channel is obtained by squaring the modulus of the complete wavefunction RFI and integrating over the hypersolid angles $\bar{\Omega}$ and the internal coordinates of the particles:

$$\text{flux out} = \int |R_{\gamma\gamma'}(\xi)|^2 \, |F_{\gamma'}(\bar{\chi}\bar{\phi}_1\bar{\phi}_2; \chi\phi_1\phi_2)|^2 \times |I(\rho\sigma\tau)|^2 \, d\bar{\Omega} \, d\Omega v_{\gamma'} \xi^3 \, d\rho \, d\sigma \, d\tau = (2\pi/k)^3 \, |\delta_{\gamma\gamma'} - S_{\gamma\gamma'}|^2, \quad (73)$$

where $v_{\gamma'}$ is the hypervelocity of the particles in the exit channel γ'. Using the definition of cross section (40) and the fact that the entrance flux is unity, the cross section for exit channel γ' can be expressed as

$$\sigma(\gamma \to \gamma') = (2\pi/k)^3 \, |\delta_{\gamma\gamma'} - S_{\gamma\gamma'}|^2 \quad (74)$$

For the case of scattering by a "central" potential (Sec. 4) the phase shifts η_λ can be related to the diagonal elements of the scattering matrix by the familiar relation

$$S_{\lambda\lambda} = e^{2i\eta_\lambda}, \quad (75)$$

where η_λ is in general a complex quantity, the real part of which is associated with elastic scattering and the imaginary part with inelastic scattering.

6. IDENTICAL PARTICLES

In the foregoing treatment of three-particle scattering we have considered distinguishable particles only. If we are to extend the case to indistinguishable particles (fermions or bosons), the wavefunctions must be properly symmetrized:

$$\psi = (6)^{-\frac{1}{2}} \sum_{\mathcal{P}} \zeta_{\mathcal{P}} \mathcal{P} \times \{\exp[i(\mathbf{p}^1 \cdot \mathbf{x}^1 + \mathbf{p}^2 \cdot \mathbf{x}^2 + \mathbf{p}^3 \cdot \mathbf{x}^3)]\}, \quad (76)$$

where $\mathbf{p}^i$ and $\mathbf{x}^i$ are the momentum and position of particle i in the laboratory system, $\mathcal{P}$ is the particle permutation operator and $\zeta_{\mathcal{P}}$ is described by

$$\zeta_{\mathcal{P}} = 1 \text{ bosons,}$$

$$\zeta_{\mathcal{P}} = \left.\begin{array}{l} -1 \text{ for an odd number of permutations} \\ +1 \text{ for an even number of permutations} \end{array}\right\} \text{fermions.}$$

If we transform to the $\xi - \pi$ coordinate system of Eq. (7), (73) then takes the form

$$\begin{aligned}\psi = \exp(i\mathbf{P}\cdot\mathbf{X})(6)^{-\frac{1}{2}}\{&\exp[i(\pi^1\cdot\xi^1 + \pi^2\cdot\xi^2)] \\ &+ \eta \exp[\tfrac{1}{2}i(\pi^1\cdot\xi^1 - \pi^2\cdot\xi^2 \\ &- \sqrt{3}\,\pi^1\cdot\xi^2 - \sqrt{3}\,\pi^2\cdot\xi^1)] \\ &+ \eta \exp[\tfrac{1}{2}i(\pi^1\cdot\xi^1 - \pi^2\cdot\xi^2 \\ &+ \sqrt{3}\,\pi^1\cdot\xi^2 + \sqrt{3}\,\pi^2\cdot\xi^1)] \\ &+ \eta \exp[i(-\pi^1\cdot\xi^1 + \pi^2\cdot\xi^2)] \\ &+ \exp[\tfrac{1}{2}i(-\pi^1\cdot\xi^1 - \pi^2\cdot\xi^2 \\ &+ \sqrt{3}\,\pi^1\cdot\xi^2 - \sqrt{3}\,\pi^2\cdot\xi^1)] \\ &+ \exp[\tfrac{1}{2}i(-\pi^1\cdot\xi^1 - \pi^2\cdot\xi^2 \\ &- \sqrt{3}\,\pi^1\cdot\xi^2 + \sqrt{3}\,\pi^2\cdot\xi^1)]\},\end{aligned} \quad (77)$$

where $\mathbf{P}$ is the momentum and $\mathbf{X}$ the position of the center of mass of the three particles; $\eta = -1$ for fermions and $+1$ for bosons. One can now compute the free-particle Green's function just as in Sec. 3, obtaining 6 terms instead of one:

$$G_k^{(+)}(\xi, \xi') = \frac{i\mu k}{4\pi} \times \left[\frac{H_1^{(1)}(kr_1)}{r_1} + \eta\frac{H_1^{(1)}(kr_2)}{r_2} + \eta\frac{H_1^{(1)}(kr_3)}{r_3} + \eta\frac{H_1^{(1)}(kr_4)}{r_4} + \frac{H_1^{(1)}(kr_5)}{r_5} + \frac{H_1^{(1)}(kr_6)}{r_6}\right], \quad (78)$$

[12] See, for example, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 519.

where the r_i are related to ξ^1 and ξ^2 as follows:

$$r_1^2 = \xi^2 + \xi'^2 - 2(\xi^1\cdot\xi'^1 + \xi^2\cdot\xi'^2),$$
$$r_2^2 = \xi^2 + \xi'^2 - (\xi^1\cdot\xi'^1 - \xi^2\cdot\xi'^2 - \sqrt{3}\,\xi^1\cdot\xi'^2 - \sqrt{3}\,\xi^2\cdot\xi'^1),$$
$$r_3^2 = \xi^2 + \xi'^2 - (\xi^1\cdot\xi'^1 - \xi^2\cdot\xi'^2 + \sqrt{3}\,\xi^1\cdot\xi'^2 - \sqrt{3}\,\xi^2\cdot\xi'^1),$$
$$r_4^2 = \xi^2 + \xi'^2 - 2(-\xi^1\cdot\xi'^1 + \xi'\cdot\xi'^2),$$
$$r_5^2 = \xi^2 + \xi'^2 - (-\xi^1\cdot\xi'^1 - \xi^2\cdot\xi'^2 + \sqrt{3}\,\xi^1\cdot\xi'^2 - \sqrt{3}\,\xi^2\cdot\xi'^1),$$
$$r_6^2 = \xi^2 + \xi'^2 - (-\xi^1\cdot\xi'^1 - \xi^2\cdot\xi'^2 - \sqrt{3}\,\xi^1\cdot\xi'^2 + \sqrt{3}\,\xi^2\cdot\xi'^1).$$

Computation of the scattering amplitude for the system leads to

$$f(\bar{\chi}\bar{\phi}_1\bar{\phi}_2, \chi\phi_1\phi_2) = \sum_{i=1}^{6} \eta^{(i)} f^{(i)}(\bar{\chi}\bar{\phi}_1\bar{\phi}_2, \chi\phi_1\phi_2), \tag{79}$$

in which $\eta^{(i)} = +1$ for $i = 1$, 5, and 6; $\eta^{(i)} = -1$ for $i = 2$, 3, and 4 in the case of fermions; $\eta^{(i)} = 1$ for bosons. The different component scattering amplitudes $f^{(i)}$ are of the same form as (29) with the coefficient α of the product $k\xi$ which appears in the exponential factor $e^{-ik\xi'\alpha}$ assuming the values α_i.

$$\alpha_1 = \cos\chi\cos\bar{\chi}\cos(\phi_1 - \bar{\phi}_1) + \sin\chi\sin\bar{\chi}\cos(\phi_2 - \bar{\phi}_2),$$
$$\alpha_2 = \tfrac{1}{2}\cos\chi\cos\bar{\chi}\cos(\phi_1 - \bar{\phi}_1) - \tfrac{1}{2}\sin\chi\sin\bar{\chi}\cos(\phi_2 - \bar{\phi}_2) - \tfrac{1}{2}\sqrt{3}\cos\chi\sin\bar{\chi}\cos(\phi_1 - \bar{\phi}_2) - \tfrac{1}{2}\sqrt{3}\sin\chi\cos\bar{\chi}\cos(\phi_2 - \bar{\phi}_1),$$
$$\alpha_3 = \tfrac{1}{2}\cos\chi\cos\bar{\chi}\cos(\phi_1 - \bar{\phi}_1) - \tfrac{1}{2}\sin\chi\sin\bar{\chi}\cos(\phi_2 - \bar{\phi}_2) + \tfrac{1}{2}\sqrt{3}\cos\chi\sin\bar{\chi}\cos(\phi_1 - \bar{\phi}_2) + \tfrac{1}{2}\sqrt{3}\sin\chi\cos\bar{\chi}\cos(\phi_2 - \bar{\phi}_1),$$
$$\alpha_4 = -\cos\chi\cos\bar{\chi}\cos(\phi_1 - \bar{\phi}_1) + \sin\chi\sin\bar{\chi}\cos(\phi_2 - \bar{\phi}_2),$$
$$\alpha_5 = -\tfrac{1}{2}\cos\chi\cos\bar{\chi}\cos(\phi_1 - \bar{\phi}_1) - \tfrac{1}{2}\sin\chi\sin\bar{\chi}\cos(\phi_2 - \bar{\phi}_2) + \tfrac{1}{2}\sqrt{3}\cos\chi\sin\bar{\chi}\cos(\phi_1 - \bar{\phi}_2) - \tfrac{1}{2}\sqrt{3}\sin\chi\cos\bar{\chi}\cos(\phi_2 - \bar{\phi}_1),$$

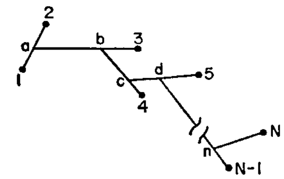

FIG. 2. Coupling of N particles with the aid of the coordinate system given in ($A1$); a denotes the center of mass of particles 1 and 2, b that of particles 1, 2, and 3. $\cdots$, n that of particles 1 to $N-1$.

$$\alpha_6 = -\tfrac{1}{2}\cos\chi\cos\bar{\chi}\cos(\phi_1 - \bar{\phi}_1) - \tfrac{1}{2}\sin\chi\sin\bar{\chi}\cos(\phi_2 - \bar{\phi}_2) - \tfrac{1}{2}\sqrt{3}\cos\chi\sin\bar{\chi}\cos(\phi_1 - \bar{\phi}_2) + \tfrac{1}{2}\sqrt{3}\sin\chi\cos\bar{\chi}\cos(\phi_2 - \bar{\phi}_1), \tag{80}$$

corresponding to the various $f^{(i)}$. The relation between the cross section and the scattering amplitude (43) can easily be shown to be correct if we define f by Eq. (79).

If only two of the particles are identical, expression (80) contains only 2 terms in the right-hand member and these are characterized by

$$\alpha_1 = \cos\chi\cos\bar{\chi}\cos(\phi_1 - \bar{\phi}_1) + \sin\chi\sin\bar{\chi}\cos(\phi_2 - \bar{\phi}_2),$$
$$\alpha_2 = -\cos\chi\cos\bar{\chi}\cos(\phi_1 - \bar{\phi}_1) + \sin\chi\sin\bar{\chi}\cos(\phi_2 - \bar{\phi}_2), \tag{81}$$

ACKNOWLEDGMENTS

The author is indebted to Drs. Felix Smith, Michael Stallybrass, and Joel Brenner for many helpful discussions.

APPENDIX A. N-BODY SCATTERING IN A PLANE

The free-particle Green's function obtained for three bodies in Sec. 3 can easily be generalized to N bodies by using a (normalized) coordinate system corresponding to the coupling scheme shown in Fig. 2.

The coordinate system is a simple extension of Smith's asymmetric one [Eq. (7)]:

$$\begin{aligned}
\xi_1^1 &= \rho\cos\chi_{N-2}\cos\chi_{N-3}\cdots\cos\chi_1\cos\phi_1,\\
\xi_2^1 &= \rho\cos\chi_{N-2}\cdots\cos\chi_1\sin\phi_1,\\
\xi_1^2 &= \rho\cos\chi_{N-2}\cdots\sin\chi_1\cos\phi_2,\\
\xi_2^2 &= \rho\cos\chi_{N-2}\cdots\sin\chi_1\sin\phi_2,\\
\xi_1^3 &= \rho\cos\chi_{N-2}\cdots\sin\chi_2\cos\phi_3,\\
\xi_2^3 &= \rho\cos\chi_{N-2}\cdots\sin\chi_2\sin\phi_3,\\
&\vdots\\
\xi_1^{N-1} &= \rho\sin\chi_{N-2}\cos\phi_{N-1},\\
\xi_2^{N-1} &= \rho\sin\chi_{N-2}\sin\phi_{N-1},
\end{aligned} \tag{A1}$$

where

$$\rho^2 = \sum_{j=1}^{N-1} [(\xi_1^j)^2 + (\xi_2^j)^2],$$

and

$$0 \le \chi_i \le \tfrac{1}{2}\pi, \qquad 0 \le \phi_i \le 2\pi.$$

The N-body plane wavefunctions in the center of mass of the assembly are of the form

$$\psi_i(\xi) = [1/(2\pi)^{N-1}] \times \exp [i(\pi^1 \cdot \xi^1 + \pi^2 \cdot \xi^2 + \cdots \pi^{N-1} \cdot \xi^{N-1})] \quad \text{(A2)}$$

and the Green's function is accordingly generalized to

$$G_{k_\alpha}^{(+)}(\xi, \xi') = \lim_{\epsilon \to +0} \frac{2\mu}{(2\pi)^{2N-2}} \times \int \frac{\exp \left[i \sum_{j=1}^{N-1} \pi^i \cdot (\xi^i - \xi'^i) \right]}{k_{\alpha^2} - k^2 + 2i\mu\epsilon} d\pi, \quad \text{(A3)}$$

where

$$k^2 = \sum_{i=1}^{N-1} |\pi^i|^2.$$

Introducing the coordinate system (A1) and integrating over the "azimuthal" angles ϕ_i, we obtain

$$G_{k_\alpha}^{(+)}(\xi, \xi') = \lim_{\epsilon \to +0} \frac{2\mu}{(2\pi)^{(N-1)}} \times \int_0^{\frac{1}{2}\pi} \cdots \int_0^{\frac{1}{2}\pi} \int_0^\infty \frac{\prod_{i=1}^{N-2} J_0\left(ka_i \sin \chi_{i-1} \prod_{j=1}^{N-2} \cos \chi_j\right)}{k_\alpha^2 - k^2 + 2i\mu\epsilon}$$
$$\times \sin \chi_1 \cos \chi_1 \cdots \sin \chi_{N-2} \times \cos^{2N-5} \chi_{N-2}\, d\chi_1 \cdots d\chi_{N-2} k^{2N-5}\, dk, \quad \text{(A4)}$$

where

$$a_j = (|\xi^j|^2 + |\xi'^j|^2 - 2\xi^j \cdot \xi'^j)^{\frac{1}{2}}$$

and $\chi_0 = \frac{1}{2}\pi$.

The integration over the magnitude of the momentum is carried out as in Sec. 2 and

$$G_k^{(+)}(\xi, \xi') = \tfrac{1}{2}i\mu \frac{(-1)^{N-2}}{(2\pi)^{N-2}} \times \left(a^{-1} \frac{\partial}{\partial a}\right)^{N-2} H_0^{(1)}(ka), \quad \text{(A5)}$$

where

$$a = \sum_{j=1}^{N-1} (a^j)^2 = |\xi - \xi'|.$$

Asymptotically, this takes the form

$$G_k^{(+)}(\xi, \xi') \sim e^{-(i\pi/4)(1+2N)} \frac{\mu k^{N-5/2}}{(2\pi)^{N-\frac{3}{2}}} \frac{e^{ika}}{a^{N-\frac{3}{2}}}, \quad \text{(A6)}$$

and the wavefunction of the system can then be expressed as

$$\psi(\xi) = \phi(\xi) + \frac{e^{i\pi(5/4 - \frac{1}{2}N)}}{\xi^{N-\frac{3}{2}}} k^{N-5/2} e^{ik\xi} f(\hat{\pi}_0, \hat{\pi}_i), \quad \text{(A7)}$$

where $f(\hat{\pi}_0, \hat{\pi}_i)$, the scattering amplitude, is written

$$f(\hat{\pi}_0, \hat{\pi}_i) = \frac{\mu}{(2\pi)^{N-\frac{3}{2}}} \int e^{-ik\xi'\alpha} V(\xi') \psi_{\pi_i}(\xi')\, d\xi', \quad \text{(A8)}$$

and

$$\alpha = \sum_{j=0}^{N-2} \sin \chi_j \sin \chi_j' \cos (\phi_{j+1} - \phi_{j+1}') \times \left[\prod_{k=j+1}^{N-2} \cos \chi_k \cos \chi_k' \right]. \quad \text{(A9)}$$

Using the N-body analogue of (40), the cross section for N particles is related to the scattering amplitude by

$$\sigma(\mathbf{k}, \mathbf{k}_0) = (k^{2N-4}/k_0)\, |f(\hat{\pi}_0, \hat{\pi}_i)|^2. \quad \text{(A10)}$$

APPENDIX B. PLANE WAVE EXPANSION

In Sec. 3, the expansion of the three-particle plane wave $\exp(i\pi \cdot \xi)$ in radial and momentum eigenfunctions proved to be a useful mathematical device:

$$\exp(i\pi \cdot \xi) = \sum_{\lambda m_1 m_2} C_{\lambda m_1 m_2} \frac{J_{\lambda+1}(k\xi)}{k\xi} \mathcal{G}^*_{\lambda m_1 m_2}(\hat{\pi}) \mathcal{G}_{\lambda m_1 m_2}(\hat{\xi}). \quad \text{(B1)}$$

If one multiplies Eq. (5) by $\mathcal{G}^*_{\lambda' m_1' m_2'}(\hat{\xi})\, \mathcal{G}_{\lambda' m_1' m_2'}(\hat{\pi})$ and integrates over all possible directions of $\hat{\pi}$ and $\hat{\xi}$, one obtains, from the orthonormality properties of the $\mathcal{G}$'s,

$$C_{\lambda m_1 m_2} \frac{J_{\lambda+1}(k\xi)}{k\xi} = \iint \exp(i\pi \cdot \xi) \mathcal{G}^*_{\lambda m_1 m_2}(\hat{\xi}) \mathcal{G}_{\lambda m_1 m_2}(\hat{\pi})\, d\Omega_\xi\, d\Omega_\pi. \quad \text{(B2)}$$

We now differentiate Eq. (B2) λ times with respect to $k\xi$ and set $k\xi = 0$; obtaining

$$C_{\lambda m_1 m_2} = (2)^{\lambda+1}(\lambda + 1)(i)^\lambda \iint [\cos \chi \cos \bar{\chi} \times \cos(\phi_1 - \bar{\phi}_1) + \sin \chi \sin \bar{\chi} \cos(\phi_2 - \bar{\phi}_2)]^\lambda$$
$$\times \mathcal{G}^*_{\lambda m_1 m_2}(\chi\phi_1\phi_2) \mathcal{G}_{\lambda m_1 m_2}(\bar{\chi}\bar{\phi}_1\bar{\phi}_2)\, d\bar{\Omega}\, d\Omega, \quad \text{(B3)}$$

where $d\Omega$ is an element of hypersolid angle $d\Omega = \frac{1}{2} \sin 2\chi\, d\chi\, d\phi_1\, d\phi_2$ $(0 \le \chi \le \frac{1}{2}\pi, 0 \le \phi_1, \phi_2 \le 2\pi)$. Integration of (B3) yields

$$
\begin{aligned}
C_{\lambda m_1 m_2} = {} & (2\pi)^2(\lambda+1)^2(i)^\lambda \lambda! \\
& \times \sum_{\nu=|m_2|}^{\lambda-|m_1|} \left[\frac{[\frac12(\lambda - m_1 + m_2)]!\,[\frac12(\lambda + m_1 - m_2)]!\,[\frac12(\lambda - m_1 m_2)]!\,[\frac12(\lambda + m_1 + m_2)]!}{[\frac12(\lambda - m_1 - \nu)]!\,[\frac12(\lambda + m_1 - \nu)]!\,[\frac12(\nu - m_2)]!\,[\frac12(\nu + m_2)]!} \right] \\
& \times \Bigg\{ \sum_{\kappa=0}^{\frac12(\lambda - m_1 - m_2)} (-1)^\kappa \{\kappa!\,(m_1+\kappa)!\,[\tfrac12(\lambda - m_1 + m_2) - \kappa]!\,[\tfrac12(\lambda - m_1 - m_2) - \kappa]!\}^{-1} \\
& \times B[\tfrac12(\lambda - m_1 - 2\kappa + \nu + 2), \tfrac12(\nu + m_1 + 2\kappa - \nu + 2)] \Bigg\}^2, \qquad \text{(B4)}
\end{aligned}
$$

where $B(r, s)$ is the beta function. It was found by inserting various allowed values of λ, m_1, and m_2 into (B4) that, in each case, $C_{\lambda m_1 m_2}$ was independent of m_1 and m_2 and equal to $(2\pi)^2(i)^\lambda$. Proof of this in the general case has so far eluded the author, however. Explicitly,

$$
e^{i\boldsymbol{\pi}\cdot\boldsymbol{\xi}} = (2\pi)^2 \sum_\lambda (i)^\lambda \times \frac{J_{\lambda+1}(k\xi)}{k\xi} \sum_{m_1 m_2} \mathcal{J}^*_{\lambda m_1 m_2}(\hat{\pi})\mathcal{J}_{\lambda m_1 m_2}(\hat{\xi}). \qquad \text{(B5)}
$$

Note added in proof. Since submission of this paper for publication, Dr. Edward Gerjuoy has pointed out to the author that many years ago Sommerfeld derived the Green's function for a many-dimensional space[13] by a simpler method than that presented here. As applied to many-particle scattering, it involves a spacial partitioning which is different from ours, i.e. Sommerfeld's computation is carried out in the space of *one* of the particles. In spite of its more complicated form, the author, however, feels that the method of computation presented in the foregoing is somewhat preferable for our purposes, particularly for the derivation of Eqs. (25) and (26).

[13] E. Gerjuoy, Ann. Phys. 5, 58 (1958); A. Sommerfeld, *Partial Differential Equations in Physics* (Academic Press Inc., New York, 1949).